Solution Set 7

1. (a) For the Adams-Bashforth method the $\sigma - \lambda$ relation is

$$\sigma^2 - \left(1 + \frac{3}{2}\lambda h\right)\sigma + \frac{1}{2}\lambda h = 0$$

Letting $\lambda = i$, and solving for σ gives

$$\sigma = \left(\frac{1}{2} + i\frac{3}{4}h\right) \pm \frac{1}{2}\left(1 - \frac{9}{4}h^2 + ih\right)^{\frac{1}{2}}$$

Let

$$r = \frac{1}{2}\left[\left(1 - \frac{9}{4}h^2\right)^2 + h^2\right]^{\frac{1}{4}} \quad \text{and} \quad \theta = \frac{1}{2}\arctan\left(\frac{h}{1 - \frac{9}{4}h^2}\right)$$

then

$$\sigma_r = \frac{1}{2} \pm r\cos\theta \quad \text{and} \quad \sigma_i = \frac{3}{4}h \pm r\sin\theta$$

The principal σ-root is had by taking the positive sign in the above equations.

(b) For the second order Runge-Kutta method the $\sigma - \lambda$ relation is

$$\sigma - 1 - \lambda h - \frac{1}{2}(\lambda h)^2 = 0$$

Letting $\lambda = i$, and solving for σ gives

$$\sigma = 1 - \frac{1}{2}h^2 + ih$$

where obviously

$$\sigma_r = 1 - \frac{1}{2}h^2 \quad \text{and} \quad \sigma_i = h$$

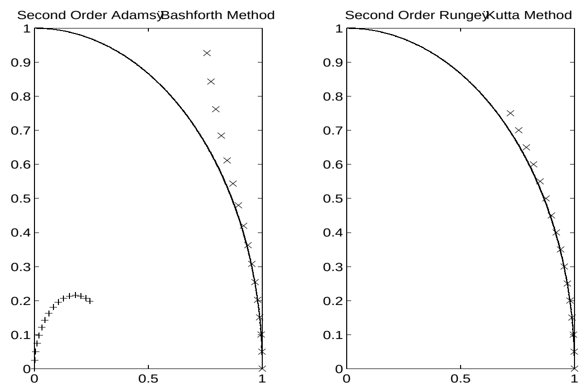
See the attached table and plots of the σ-roots for the two methods. An abbrieviated version of the matlab code used to generate the table and plots is included.

```
% Part of prob_6_95_ans   AE296  Spring 1995
%
clear;
h       = 0:0.05:0.75;                    % time steps
uc      = exp(i*linspace(0,pi/2,100));    % unit circle
a       = 1/2 + 3*i*h/4;
b       = sqrt(1-9*h.^2/4+i*h)/2;
absig1  = a+b;                            % Adams-Bashforth principal root
absig2  = a-b;                            % Adams-Bashforth spruious  root
rksig1  = 1-h.^2/2+i*h;                   % Runge-Kutta principal root

% tabular output

f  = fopen('hmw6.tab','w');
ta = [h;real(absig1);imag(absig1);abs(absig1);  ...
    real(absig2);imag(absig2);abs(absig2);  ...
    real(rksig1);imag(rksig1);abs(rksig1)];
fprintf(f,'%14.2f%11.6f%11.6f%11.6f%11.6f%11.6f%11.6f%11.6f%11.6f%11.6f\n',ta);
```

```
%plots
clf
subplot(1,2,1),plot(uc,'-'),hold on,plot(absig1,'rx'),plot(absig2,'g+');
title('Second Order Adams-Bashforth Method');
subplot(1,2,2),plot(uc,'-'),hold on,plot(rksig1,'rx');
title('Second Order Runge-Kutta Method');
```



2. For $h = 0.2$ the real and imaginary parts of the principal σ-roots for the second order Adams-Bashforth and second order Runge-Kutta methods are, respectively,

$$\begin{aligned} \sigma_r &= 0.979807, \qquad \sigma_i = 0.202104 \\ \sigma_r &= 0.980000, \qquad \sigma_i = 0.200000 \end{aligned}$$

   (a) After N time steps, the global error in amplitude for the biconvection model is given as

$$Er_A \equiv 1 - \left(\sigma_r^2 + \sigma_i^2\right)^{\frac{N}{2}}$$

   For the second order Adams-Bashforth and second order Runge-Kutta methods these amplitude errors after 100 time steps are, respectively,

$$\begin{aligned} Er_A &= -0.0443408 \\ Er_A &= -0.0201973 \end{aligned}$$

   (b) After N time steps, the global error in phase for the biconvection model is given as

$$Er_\omega \equiv N\left[\omega h - \arctan\left(\frac{\sigma_i}{\sigma_r}\right)\right]$$

   For the second order Adams-Bashforth and second order Runge-Kutta methods these phase errors for $\omega = 1$ after 100 time steps are, respectively,

$$\begin{aligned} Er_\omega &= -0.341657 \\ Er_\omega &= -0.131710 \end{aligned}$$

   (Don't bother doing a Taylor series expansion of the arctan term—here a numerical value is wanted, not the time step size dependence.)

The second order Runge-Kutta method has lower amplitude and phase errors, so for the pure convection problem it is the better method.

3. For the PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

The following difference schemes are used for the spacial derivatives

$$(\delta_x u)_j = \frac{1}{\Delta x}\left(u_{j+1} - u_j\right)$$
$$(\delta_{xx} u)_j = \frac{1}{\Delta x^2}\left(u_{j+1} - 2u_j + u_{j-1}\right)$$

(a) In banded matrix operator notation the resulting ODE is

$$\frac{d\mathbf{u}}{dt} = B\left(\frac{\nu}{\Delta x^2}, \frac{a}{\Delta x} - 2\frac{\nu}{\Delta x^2}, \frac{\nu}{\Delta x^2} - \frac{a}{\Delta x}\right)\mathbf{u} + (bc)$$

The eigenvalues for a simple tridiagonal matrix with constant scalar elements along each diagonal are given in equation (B1.2) of the lecture notes. For this problem the eigenvalues are

$$\lambda_m = \frac{a}{\Delta x} - 2\frac{\nu}{\Delta x^2} + 2\sqrt{\frac{\nu}{\Delta x^2}\left(\frac{\nu}{\Delta x^2} - \frac{a}{\Delta x}\right)}\cos\left(\frac{m\pi}{M+1}\right) \qquad m = 1, 2, \ldots, M$$

(b) A method is inherently stable when $\Re(\lambda_m) \leq 0$ for all m. For this method, this leads to the following stability constraint

$$a \leq 2\frac{\nu}{\Delta x}$$

4. Substituting $u = e^{\alpha t + ikx}$ into the Lax-Wendroff scheme given by

$$u_j^{n+1} = u_j^n - \frac{1}{2}C_n\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}C_n^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

yields the amplification factor $\sigma = e^{\alpha\Delta t}$

$$\sigma = 1 - \frac{1}{2}C_n\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) + \frac{1}{2}C_n^2\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$
$$= 1 - iC_n\sin k\Delta x + C_n^2(\cos k\Delta x - 1)$$

It follows that

$$|\sigma|^2 = 1 + 2C_n^2(\cos k\Delta x - 1) + C_n^4(\cos k\Delta x - 1)^2 + C_n^2\sin^2 k\Delta x$$
$$= 1 + C_n^2\left(C_n^2 - 1\right)\left(\cos k\Delta x - 1\right)^2$$

and stability requires that $|\sigma| \leq 1$ so that the condition on C_n becomes

$$C_n^2\left(C_n^2 - 1\right)\left(\cos k\Delta x - 1\right)^2 \leq 0$$
$$C_n^2 \leq 1$$